

A REMARK ON EINSTEIN WARPED PRODUCTS

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ABSTRACT. We prove triviality results for Einstein warped products with non-compact bases. These extend previous work by D.-S. Kim and Y.-H. Kim. The proofs, from the viewpoint of “quasi-Einstein manifolds” introduced by J. Case, Y.-S. Shu and G. Wei, rely on maximum principles at infinity and Liouville-type theorems.

1. INTRODUCTION

The main purpose of this note is to prove the following triviality result for Einstein warped products which extends, to the case of non-compact bases, a recent theorem by D.-S. Kim and Y.-H. Kim, [7].

Theorem 1. *Let $N^{n+m} = M^n \times_u F^m$, $m > 1$, be a complete Einstein warped product with non-positive scalar curvature $^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{m}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F . Then N is simply a Riemannian product if either one of the following further conditions is satisfied:*

- (a) *f has a local minimum.*
- (b) *the base manifold M is complete and non-compact, the warping function satisfies $\int_M |f|^p e^{-\frac{f}{m}} d\text{vol} < +\infty$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.*

Note that, in case M is compact, from the point (a) we recover the main result in [7].

Our proof of Theorem 1 will rely on the link between Einstein warped product metrics and the so called “quasi-Einstein metrics” recently introduced by J. Case, Y.-S. Shu and G. Wei, [3]. In the spirit of [13], i.e. using methods from stochastic analysis and L^p -Liouville type theorems, we shall prove scalar curvature estimates and triviality results for a complete quasi-Einstein manifold that largely extend previous theorems in [3]. Whence, the main theorem will follow immediately.

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In a final section, using similar techniques, we extend another triviality result for Einstein warped products obtained in the very recent [2]. A non-existence result is also discussed.

2. QUASI-EINSTEIN MANIFOLDS

Consider the weighted manifold $(M^n, g_M, e^{-f} d\text{vol})$, where M is a complete n -dimensional Riemannian manifold, f is a smooth real valued function on M and $d\text{vol}$ is the Riemannian volume density on M . A natural extension of the Ricci tensor to weighted manifolds is the m -Bakry-Emery Ricci tensor

$$\text{Ric}_f^m = \text{Ric} + \text{Hess}f - \frac{1}{m} df \otimes df, \quad \text{for } 0 < m \leq \infty.$$

When f is constant, this is the usual Ricci tensor and when $m = \infty$ this is the Ricci Bakry-Emery tensor Ric_f . We call a metric m -quasi-Einstein if the m -Bakry-Emery Ricci tensor satisfies the equation

$$(1) \quad \text{Ric}_f^m = \lambda g_M,$$

for some $\lambda \in \mathbb{R}$. This equation is especially interesting in that when $m = \infty$ it is exactly the gradient Ricci soliton equation. When f is constant, it gives the Einstein equation and we call the quasi-Einstein metric trivial. When m is a positive integer, it corresponds to warped product Einstein metrics.

Indeed, in [3], elaborating on [7], it is observed the following characterization of quasi-Einstein metrics.

Theorem 2. *Let $M^n \times_u F^m$ be an Einstein warped product with Einstein constant λ , warping function $u = e^{-\frac{f}{m}}$ and Einstein fibre F^m . Then the weighted manifold $(M^n, g_M, e^{-f} d\text{vol})$ satisfies the quasi-Einstein equation (1). Furthermore the Einstein constant μ of the fibre satisfies*

$$(2) \quad \Delta f - |\nabla f|^2 = m\lambda - m\mu e^{\frac{2}{m}f}.$$

Conversely if the weighted manifold $(M^n, g_M, e^{-f} d\text{vol})$ satisfies (1), then f satisfies (2) for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{n+m} = M^n \times_u F^m$, with $u = e^{-\frac{f}{m}}$ and Einstein fibre F with ${}^F\text{Ric} = \mu g_F$. Then N is Einstein with ${}^N\text{Ric} = \lambda g_N$.

3. SCALAR CURVATURE ESTIMATES

In this section, in the same spirit of Theorem 3 of [13], we generalize the scalar curvature estimates in Proposition 3.6 of [3] to quasi-Einstein manifolds with non-constant scalar curvature. Possible rigidity at the endpoints is also discussed.

Theorem 3. *Let $(M^n, g_M, e^{-f} d\text{vol})$ be a geodesically complete m -quasi-Einstein manifold, $1 < m < +\infty$, with scalar curvature S and let $S_* = \inf_M S$.*

(a) *If $\lambda > 0$, then M is compact and*

$$(3) \quad \frac{n(n-1)}{m+n-1} \lambda < S_* \leq n\lambda.$$

Moreover $S_ \neq n\lambda$ unless M is Einstein.*

(b) *If $\lambda = 0$ and $\inf_M f = f_* > -\infty$ then $S_* = 0$. Moreover, either $S > 0$ or $S(x) \equiv 0$. In this latter case, either f is constant (and M is trivial) or M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ where Σ is a Ricci-flat, totally geodesic hypersurface.*

(c) *If $\lambda < 0$ and $\inf_M f = f_* > -\infty$, then*

$$(4) \quad n\lambda \leq S_* \leq \frac{n(n-1)}{m+n-1} \lambda$$

and $S(x) > n\lambda$ unless M is Einstein.

The proof of Theorem 3 will require the following formula obtained in [3], which generalizes to the case $m < +\infty$ similar formulas for Ricci solitons ($m = +\infty$) obtained previously by P. Petersen and W. Wylie, [12]. Following the terminology introduced in [11], the f -Laplacian on the weighted manifold $(M, g_M, e^{-f} d\text{vol})$ is the diffusion type operator defined by $\Delta_f u = e^f \text{div}(e^{-f} \nabla u)$. It is clearly a symmetric operator on $L^2(M, e^{-f} d\text{vol})$.

Lemma 4. *Let $\text{Ric}_f^m = \lambda g_M$, for some $\lambda \in \mathbb{R}$ and $m < +\infty$. Set $\tilde{f} = \frac{m+2}{m} f$. Then*

$$(5) \quad \frac{1}{2} \Delta_{\tilde{f}} S = -\frac{m-1}{m} |\text{Ric} - \frac{1}{n} S g_M|^2 - \frac{m+n-1}{mn} (S - n\lambda) (S - \frac{n(n-1)}{m+n-1} \lambda).$$

Proof (of Theorem 3). First of all, we show that $\inf_M S > -\infty$. According to Qian version of Myers' theorem this is obvious if $\lambda > 0$ because M is compact, see also the Appendix. In the general case $\lambda \in \mathbb{R}$ we proceed as follows. Since

$$-\left| \text{Ric} - \frac{1}{n} S g_M \right|^2 = -|\text{Ric}|^2 + \frac{S^2}{n},$$

from (5) we obtain

$$(6) \quad \begin{aligned} \frac{1}{2} \Delta_{\tilde{f}} S &= -\frac{m-1}{m} |\text{Ric}|^2 - \frac{1}{m} S^2 + \frac{m+2n-2}{m} S \lambda - \frac{n(n-1)}{m} \lambda^2. \\ &\leq -\frac{1}{m} S^2 + \frac{m+2n-2}{m} \lambda S. \end{aligned}$$

Let $S_-(x) = \max\{-S(x), 0\}$. Then

$$(7) \quad \Delta_{\tilde{f}} S_- \geq \frac{2}{m} S_-^2 + \frac{2(m+2n-2)}{m} \lambda S_-.$$

Observe now that from Qian's estimates of weighted volumes ([14], see also section 2 in [8] and references therein), since $\text{vol}_{\tilde{f}}(B_r) \leq e^{-\frac{2}{m} \tilde{f}^*} \text{vol}_f(B_r)$, we can apply the “a-priori” estimate in Theorem 12 of [13] to inequality (7) on the complete weighted manifold $(M, g_M, e^{-\tilde{f}} d\text{vol})$ and we obtain that S_- is bounded from above, or equivalently, $S_* = \inf_M S > -\infty$. Again from the volume estimates in [14] and by Theorem 9 in [13] applied to $(M, g_M, e^{-\tilde{f}} d\text{vol})$, the weak maximum principle at infinity for the \tilde{f} -laplacian holds on M . This produces a sequence $\{x_k\}$ such that $\Delta_{\tilde{f}} S(x_k) \geq -\frac{1}{k}$ and $S(x_k) \rightarrow S_*$. Taking the \liminf in (5) along $\{x_k\}$ shows that, for $m > 1$,

$$(8) \quad 0 \leq -\frac{m+n-1}{mn} (S_* - n\lambda) (S_* - \frac{n(n-1)}{m+n-1} \lambda).$$

We now distinguish three cases.

(a) Assume $\lambda > 0$, so that M is compact. Equation (8) yields $\frac{n(n-1)}{m+n-1} \lambda \leq S_* \leq n\lambda$. Assume now that $S_* = n\lambda > 0$. Then $S \geq n\lambda \geq \frac{n(n-1)}{m+n-1} \lambda$ and from (5) we get

$$\frac{1}{2} \Delta_{\tilde{f}} S \leq -\frac{m+n-1}{mn} (S - n\lambda) (S - \frac{n(n-1)}{m+n-1} \lambda) \leq 0.$$

Since M is compact, S must be constant. Hence $S = S_* = n\lambda$. Substituting in (5) we obtain that $\text{Ric} = \frac{1}{n} S g_M$ and thus that M is Einstein.

Now we show that $S_* > \frac{n(n-1)}{m+n-1} \lambda$. Indeed, suppose that S attains its minimum $\frac{n(n-1)}{m+n-1} \lambda$. Since the non-negative function $v(x) = S(x) - \frac{n(n-1)}{m+n-1} \lambda$ satisfies

$$\frac{1}{2} \Delta_{\tilde{f}} v \leq -\frac{m+n-1}{mn} v^2 + \lambda v \leq +\lambda v,$$

and v attains its minimum $v(x_0) = 0$, it follows from the minimum principle, (see p. 35 in [6]), that v vanishes identically. Hence $S \equiv \frac{n(n-1)}{m+n-1} \lambda$ is constant and, substituting in (5), we get that M is Einstein with

$$\text{Ric} = \frac{n-1}{m+n-1} \lambda g_M.$$

Using this information into (1) we obtain that

$$\text{Hess}(f) = \frac{1}{m} |\nabla f|^2 + \frac{m}{m+n-1} \lambda g_M > 0.$$

But this is clearly impossible because M is compact.

(b) Assume $\lambda = 0$. From (8) we conclude that $S_* = 0$. Note that, according to (5), $\Delta_{\hat{f}} S \leq 0$. Therefore, by the minimum principle, either $S(x) > 0$ on M or $S(x) \equiv 0$. In this latter case, substituting in (5), we obtain that M is Ricci flat and the m -quasi Einstein equation reads $\text{Hess}(f) - \frac{1}{m} df \otimes df = 0$. Therefore, either f is constant and M is Einstein, or the non constant function $u = e^{-\frac{f}{m}}$ satisfies $\text{Hess}(u) = 0$. A Cheeger-Gromoll type argument now shows that M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ along the Ricci flat, totally geodesic hypersurface Σ of M .

(c) Assume $\lambda < 0$. From (8) we deduce that $n\lambda \leq S_* \leq \frac{n(n-1)}{m+n-1}\lambda$. Suppose that $S(x_0) = n\lambda < 0$ for some $x_0 \in M$. Since the non-negative function $w(x) = S(x) - n\lambda$ satisfies

$$\frac{1}{2}\Delta_{\hat{f}} w \leq -\frac{m+n-1}{mn}w^2 - \lambda w \leq -\lambda w,$$

and w attains its minimum $w(x_0) = 0$, it follows from the minimum principle that w vanishes identically. Hence $S \equiv n\lambda$ is constant and substituting in (5) we get that M is Einstein. \square

4. TRIVIALITY RESULTS UNDER L^p CONDITIONS

It is well known that steady or expanding compact Ricci solitons are necessarily trivial. The same result is proven in [7] for quasi-Einstein metrics on compact manifolds with finite m . For Ricci solitons a generalization to the complete non-compact setting is obtained in [13].

In this section using the scalar curvature estimates of Theorem 3, we get triviality for (not necessarily compact) quasi-Einstein metrics with $m < +\infty$, $\lambda \leq 0$.

Theorem 5. *Let $(M^n, g_M, e^{-f} d\text{vol})$ be a geodesically complete non-compact m -quasi-Einstein manifold, $1 \leq m < +\infty$. If the quasi-Einstein constant λ is non-positive and f satisfies, for some $1 < p < +\infty$,*

$$(9) \quad f \in L^p(M, e^{-\frac{f}{m}} d\text{vol}),$$

and $\inf_M f = f_ > -\infty$, then either $f \equiv \text{const} \leq 0$ and M is Einstein or $f > 0$.*

Proof. (of Theorem 5) Tracing (1) and letting $\hat{f} = \frac{1}{m}f$ we have that

$$(10) \quad \Delta_{\hat{f}} f = n\lambda - S.$$

Since $\lambda \leq 0$ and $f_* > -\infty$, from (4) of Theorem 3 we obtain that $\Delta_{\hat{f}} f \leq 0$.

Applying Theorem 14 in [13] to $f_- = \max\{-f, 0\} \in L^p(M, e^{-\hat{f}} d\text{vol})$, gives that f_- is constant. Hence, if there exists a point $x_0 \in M$ such that $f(x_0) \leq 0$ then $f \equiv f(x_0) \leq 0$. \square

Remark 6. From the proof it follows that if either M is compact or f attains its absolute minimum then $f \equiv \text{const}$. Actually, it was pointed out to us by Dezhong Chen that the same conclusion holds if we merely assume that f attains a local minimum at some point $x_0 \in M$. Indeed the following proposition holds.

Proposition 7. *Let $(M, g_M, e^{-f} \text{dvol})$ be a geodesically complete non-compact m -quasi-Einstein manifold, $1 < m < +\infty$. If the quasi-Einstein constant λ is non positive and f satisfies $f_* > -\infty$, then any local minimum of f is actually an absolute minimum.*

Proof. Assume that f attains a local minimum $x_0 \in M$. Evaluating (10) at x_0 , we get

$$S(x_0) \leq n\lambda.$$

Hence, since $\lambda \leq 0$, by Theorem 3, M is Einstein and S is identically $n\lambda$. Thus the quasi-Einstein equation (1) reads

$$(11) \quad \text{Hess}(f) = \frac{1}{m} df \otimes df.$$

In particular $\text{Hess}(f)$ is positive semi-definite on M and this implies the thesis. \square

5. PROOF OF THE MAIN THEOREM

Putting together the results of the previous sections we easily obtain a proof of Theorem 1.

Indeed, according to Theorem 2, M is quasi-Einstein. Statement (a) follows immediately from Remark 6 and Proposition 7. In case (b), since $(n+m)\lambda = {}^N S \leq 0$, we get by Theorem 5 that f , and so u , is a constant function.

6. OTHER TRIVIALITY RESULTS

Another triviality result for Einstein warped products has been obtained by J. Case in [2].

Theorem 8. *(Case) Let $N^{n+m} = M^n \times_u F^m$ be a complete warped product with warping function $u(x) = e^{-\frac{f(x)}{m}}$, scalar curvature ${}^N S \geq 0$ and complete Einstein fibre F . Then N is simply a Riemannian product provided the base manifold M is complete and the scalar curvature of F satisfies ${}^F S \leq 0$.*

In the following theorem we obtain the same conclusion in case the fibers have non-negative scalar curvature, up to assume an integrability condition on the warping function u . We observe that non-trivial examples with ${}^N S \leq 0$ and ${}^F S \geq 0$ are constructed in ([1], 9.118). Thus the integrability assumption is necessary.

Theorem 9. *Let $N^{n+m} = M^n \times_u F^m$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{m}}$, scalar curvature $^N S \leq 0$, and complete Einstein fibre F . Then N is simply a Riemannian product provided the base manifold M is complete, the warping function satisfies $\int_M e^{-(\frac{p+m}{m})f} d\text{vol} < +\infty$ for some $1 < p < +\infty$, and the scalar curvature of F satisfies $^F S \geq 0$. In this case M and F are Ricci flat and M is compact.*

Combining Theorem 8 and Theorem 9 immediately gives the following

Corollary 10. *Let N be a complete Ricci flat warped product with complete Einstein fibre F and warping function $u(x) = e^{-\frac{f(x)}{m}}$ satisfying $u \in L^p(M, e^{-f} d\text{vol})$, for some $1 < p < +\infty$. Then N is simply a Riemannian product.*

Proof. (of Theorem 9) Just observe that computing the f -laplacian of u and using (2) one obtains the following equation

$$(12) \quad \Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{m^2} |\nabla f|^2.$$

Thus, in our assumptions, we obtain that $\Delta_f u \geq 0$. Since $0 < u \in L^p(M, e^{-f} d\text{vol})$, by Theorem 14 in [13], we obtain the constancy of u . Up to a rescaling of the metric of F we can suppose $u = 1$.

Now, since the Riemannian product $M \times F$ is Einstein, both M and F are Einstein manifolds with the same Einstein constant. In particular, $^M S$ and $^F S$ have the same sign. By our assumption on the signs of $^N S$ and $^F S$ we thus obtain that both M and F are Ricci flat. Finally, since u (and thus f) is constant, from the integrability condition we obtain that $\text{vol}(M) < +\infty$. Thus, by a result of Calabi-Yau, [Y], we obtain that M must be compact. \square

We end this section with a non-existence result. Recall that by the volume estimates in [14] and Theorem 9 in [13] the weak maximum principle for the f -laplacian holds on $(M, g_M, e^{-f} d\text{vol})$ provided $\text{Ric}_f^m = \lambda g_M$ for some $\lambda \in \mathbb{R}$, $m < +\infty$.

Theorem 11. *There is no complete Einstein warped product $N = M^n \times_u F^m$ with warping function $u = e^{-\frac{f}{m}} \in L^\infty(M)$, scalar curvature $^N S < 0$ and Einstein fibre F with $^F S \geq 0$.*

Proof. Since $m\mu = ^F S \geq 0$, from (12), we have that

$$(13) \quad \Delta_f u \geq -u\lambda.$$

Since, by assumption, u satisfies $\sup_M u = u^* < +\infty$, by the weak maximum principle at infinity for the f -laplacian, there exists a sequence $\{x_k\} \subset M$ along which $u(x_k) \geq u^* - \frac{1}{k}$ and $\Delta_f u(x_k) \leq \frac{1}{k}$. Thus evaluating (13) along $\{x_k\}$ and taking the limit as $k \rightarrow +\infty$ we obtain that $\lambda u^* \geq 0$ and since $u^* > 0$ we cannot have $\lambda < 0$. \square

APPENDIX

An extension of Myers' theorem to weighted manifolds with a positive lower bound on the m -Bakry-Emery Ricci tensor (m finite) is obtained by Qian in [14]. For generalizations of Myers' theorem in a different direction see [10].

In this section we extend Qian theorem by allowing some negativity of the m -Bakry-Emery Ricci tensor. The starting point of our considerations is the following Bochner formula for the m -Bakry-Emery Ricci tensor; see e.g. [15].

Let $u : M^n \rightarrow \mathbb{R}$ be a smooth function on a complete weighted manifold $(M^n, g_M, e^{-f} d\text{vol})$ then

$$(14) \quad \frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess}(u)|^2 + g_M(\nabla u, \nabla \Delta_f u) + \text{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m} |g_M(\nabla f, \nabla u)|^2.$$

Using this formula one obtains the following generalization of a well-known lemma which estimate the integral of Ricci along geodesics. The proof is modelled on [14].

Lemma 12. *Let $(M^n, g_M, e^{-f} d\text{vol})$ be a complete weighted manifold, and consider the m -Bakry-Emery Ricci tensor Ric_f^m for m finite. Fix $o \in M$ and let $r(x) = \text{dist}(x, o)$. For any point $q \in M$, let $\gamma_q : [0, r(q)] \rightarrow M$ be a minimizing geodesic from o to q such that $|\dot{\gamma}_q| = 1$. If $h \in \text{Lip}_{\text{loc}}(\mathbb{R})$ is such that $h(0) = h(r(q)) = 0$, then for every $q \in M$, it holds*

$$(15) \quad 0 \leq \int_0^{r(q)} (m+n-1) (h')^2 ds - \int_0^{r(q)} h^2 \text{Ric}_f^m(\dot{\gamma}_q, \dot{\gamma}_q) ds.$$

Proof. Fix a point $q \notin \text{cut}(o)$. Straightforward computations show that

$$(16) \quad \frac{(\Delta_f r)^2}{m+n-1} \leq \frac{(\Delta r)^2}{n-1} + \frac{|g_M(\nabla f, \nabla r)|^2}{m},$$

$$(17) \quad |\text{Hess}(r)|^2 \geq \frac{(\Delta r)^2}{n-1}.$$

Using (16) and (17), from the Bochner formula (14) applied to the distance function $r(x)$ we obtain that

$$0 \geq \frac{(\Delta_f r)^2}{m+n-1} + g_M(\nabla r, \nabla \Delta_f r) + \text{Ric}_f^m(\nabla r, \nabla r).$$

Evaluating this along a minimizing geodesic γ_q such that $|\dot{\gamma}_q| = 1$, we get

$$(18) \quad 0 \geq \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} + \frac{d}{ds}(\Delta_f(r \circ \gamma_q)) + \text{Ric}_f^m(\dot{\gamma}_q, \dot{\gamma}_q).$$

If $h \in Lip_{loc}(\mathbb{R})$, $h \geq 0$, $h(0) = 0$, multiplying (18) by h^2 and integrating on $[0, t]$, we obtain

$$0 \geq \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} ds + \int_0^t \frac{d}{ds} (\Delta_f r \circ \gamma_q) h^2 + \int_0^t h^2 Ric_f^m(\dot{\gamma}_q, \dot{\gamma}_q).$$

Since $(\Delta_f r \circ \gamma_q) h^2 \rightarrow 0$ as $r \rightarrow 0$, integrating by parts we have that

$$\begin{aligned} 0 &\geq \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} ds + h^2(t)(\Delta_f r \circ \gamma_q)(t) \\ &\quad - 2 \int_0^t h h' (\Delta_f r \circ \gamma_q) ds + \int_0^t h^2 Ric_f^m(\dot{\gamma}_q, \dot{\gamma}_q) ds. \end{aligned}$$

Since

$$-2h h' (\Delta_f r \circ \gamma_q) \geq \frac{-h^2 (\Delta_f r \circ \gamma_q)^2}{m+n-1} - (m+n-1)(h')^2,$$

we deduce that

$$0 \geq h^2(t)(\Delta_f r \circ \gamma_q) - \int_0^t (m+n-1)(h')^2 ds + \int_0^t Ric_f^m(\dot{\gamma}_q, \dot{\gamma}_q) h^2 ds$$

Thus, taking $t = r(q)$ and choosing h such that $h^2(r(q)) = 0$, we get (15) for $q \notin cut(o)$. To treat the general case one can use the Calabi trick. Namely suppose that $q \in cut(o)$. Translating the origin o to $o_\epsilon = \gamma_q(\epsilon)$ so that $q \notin cut(o_\epsilon)$, using the triangle inequality and, finally, taking the limit as $\epsilon \rightarrow 0$, one checks that (15) holds also in this case. \square

From Lemma 12 some Myers' type results can be proven. Here we state the following which generalizes a theorem of G. J. Galloway, [4].

Theorem 13. *Let $(M^n, g_M, e^{-f} dvol)$ be a complete weighted manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from p to q parameterized by arc length. Suppose that there exist constants c and $G \geq 0$ such that for each pair of points p, q it holds*

$$Ric_f^m(\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q})|_{\gamma_{p,q}(t)} \geq (m+n-1) \left[c^2 + \frac{d}{dt} (g \circ \gamma_{p,q}) \right],$$

for some $C^1(M)$ function g satisfying $\sup_M |g| \leq G$, $m < +\infty$. Then M is compact and

$$(19) \quad \text{diam}(M) \leq \frac{1}{c} \left[\frac{2G}{c} + \sqrt{\frac{4G^2}{c^2} + \pi^2} \right].$$

Proof. Define L to be the length of $\gamma_{p,q}$ between p and q and set $h(t) := \sin(\frac{\pi}{L}t)$. Compute

$$\int_0^L h^2(t)dt = \int_0^L \sin^2(\frac{\pi}{L}t)dt = \frac{L}{2}; \quad \int_0^L h'^2(t)dt = \frac{\pi^2}{L^2} \int_0^L \cos^2(\frac{\pi}{L}t)dt = \frac{\pi^2}{2L}.$$

Then, applying Lemma 12, we have

$$\begin{aligned} (20) \quad \frac{\pi^2(m+n-1)}{2L} &= \int_0^L (m+n-1) h'^2 \geq \int_0^L h^2 Ric_f^m(\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q})|_{\gamma_{p,q}} ds \\ &\geq c^2(m+n-1) \int_0^L h^2 + (m+n-1) \int_0^L h^2 \frac{d}{dt}(g \circ \gamma_{p,q}) \\ &= \frac{c^2(m+n-1)L}{2} + (m+n-1)h^2 g(\gamma_{p,q})|_0^L \\ &\quad - (m+n-1) \left[\int_0^{\frac{L}{2}} \left(\frac{d}{dt} h^2 \right) (g \circ \gamma_{p,q}) + \int_{\frac{L}{2}}^L \left(\frac{d}{dt} h^2 \right) (g \circ \gamma_{p,q}) \right] \\ &\geq \frac{c^2(m+n-1)L}{2} - (m+n-1)G \left[\int_0^{\frac{L}{2}} \left(\frac{d}{dt} h^2 \right) + \int_{\frac{L}{2}}^L \left| \frac{d}{dt} h^2 \right| \right] \\ &\geq \frac{c^2(m+n-1)L}{2} - 2(m+n-1)G \end{aligned}$$

Finally, this latter can be written as

$$c^2 L^2 - 4GL - \pi^2 \leq 0,$$

which in turn implies (19), because p and q are arbitrary. \square

Reasoning as in the classical case, ([5], [9]) the validity of (15) and an integration by parts shows that the compactness of M depends on the behavior, and on the position of the zeros, of the solution of the differential equation along minimizing geodesics

$$(21) \quad -h''(t) - \frac{Ric_f^m(\dot{\gamma}, \dot{\gamma})}{m+n-1} h(t) = 0$$

We are thus reduced to find sufficient condition on Ric_f^m for which solutions of the differential equation (21) have a first zero at finite time. Minor changes to the proofs of the results contained in [9] lead to similar compactness results in the weighted setting. In particular we state the following theorem in which a Myers' type conclusion is obtained assuming a nonpositive lower bound on Ric_f^m .

Theorem 14. *Let $\text{Ric}_f^m \geq -(m+n-1)B^2$, for some constant $B \geq 0$, $m < +\infty$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \rightarrow M$ parameterized by arc length, with $\gamma(0) = q$, it holds either*

$$(22) \quad \int_a^b t \frac{\text{Ric}_f^m(\dot{\gamma}, \dot{\gamma})}{m+n-1} dt > B \left\{ b + a \frac{e^{2Ba} + 1}{e^{2Ba} - 1} \right\} + \frac{1}{4} \log \left(\frac{b}{a} \right).$$

or

$$(23) \quad \int_a^b t^\alpha \frac{\text{Ric}_f^m(\dot{\gamma}, \dot{\gamma})}{m+n-1}(t) dt > B \left\{ b^\alpha + a^\alpha \frac{e^{2Ba} + 1}{e^{2Ba} - 1} \right\} + \frac{\alpha^2}{4(1-\alpha)} \{a^{\alpha-1} - b^{\alpha-1}\}$$

for some $0 < a < b$ and $\alpha \neq 1$. Then M is compact.

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REFERENCES

- [1] A. L. Besse, *Einstein manifolds*. Springer-Verlag, Berlin-Heidelberg (1987).
- [2] J. Case, *On the nonexistence of quasi-Einstein metrics*. To appear in Pacific J. Math. arXiv:0902.2226v3.
- [3] J. Case, Y.-S. Shu, G. Wei, *Rigidity of Quasi-Einstein metrics*. arXiv:0805.3132v1.
- [4] G. J. Galloway, *A generalization of Myers' theorem and an application to relativistic cosmology*. J. Differential Geom. 14 (1979), no. 1, 105–116 (1980).
- [5] G. J. Galloway, *Compactness criteria for Riemannian manifolds*. Proc. Amer. Math. Soc. 84 (1982), no. 1, 106–110.
- [6] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Verlag, Berlin (1983).
- [7] D.-S. Kim, Y. H. Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*. Proc. Amer. Math. Soc. 131 (2003), no. 8, 2573–2576.
- [8] L. Mari, M. Rigoli, A. G. Setti, *Keller-Osserman conditions for diffusion-type operators on Riemannian manifolds*. Jour. Funct. Anal. **258** (2010), 665–712.
- [9] P. Mastroia, M. Rimoldi and G. Veronelli, *Myers' type theorems and some related oscillation results*. arXiv:1002.2076.
- [10] F. Morgan, *Myers' theorem with density*. Kodai Math. J. **29** (2006), n. 3, 455–461.
- [11] P. Petersen, W. Wylie, *On the classification of gradient Ricci solitons*. arXiv:0712.1298.
- [12] P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*. Pacific J. Math. **241** (2009), no. 2, 329–345.
- [13] S. Pigola, M. Rimoldi, A.G. Setti, *Remarks on non-compact gradient Ricci solitons*. To appear in Math. Z. arXiv:0905.2868v3.
- [14] Z. Qian, *Estimates for the weighted volumes and applications*. Quart. J. Math. Oxford Ser. (2) **48** (1997), no.190, pp. 235–242.

- [15] A.G. Setti, *Eigenvalue estimates for the weighted Laplacian on a Riemannian manifold*. Rend. Sem. Mat. Univ. Padova **100** (1998) 27–55.
- [16] S.T. Yau, *Some function theoretic properties of complete Riemannian manifolds and their applications to geometry*. Indiana Univ. Math. J. **25** (1976), 659–670.

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